On the amplification of diffusion on piecewise linear potentials by direct current

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Abstract

The diffusive motion of overdamped Brownian particles in tilted piecewise linear potentials is considered. It is shown that the enhancement of diffusion coefficient by an external static force is quite sensitive to the symmetry of periodic potential. Another new effect found is that the factor of randomness as a function of the tilting force exhibits a plateau-like behaviour in the region of low temperatures.

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Various aspects of the dynamics of Brownian particles in external periodic potentials have been a subject of growing interest in the last few years [1-3]. The interplay of noise and nonlinear dynamics produces a rich variety of remarkable physical effects, for example, the dependence of the (direction of) particle current on the statistical properties of the nonequilibrium fluctuations which induce the current [1,4-6]. The present paper addresses the interrelationship between the current and diffusion of the Brownian particles on tilted simple sawtooth potentials. In the case of thermal equilibrium, the overdamped force-free thermal diffusion of mutually non-interacting Brownian particles is always reduced when a additional periodic potential is applied to the system, as corresponding localization of the particles occurs [7]. Therefore one tends to think that a qualitatively similar behaviour is valid at least for some time-independent non-equilibrium systems. However, recently it was discovered that the opposite is the case: in the seminal papers [8, 9] it

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was demonstrated that a giant enhancement of diffusion will take place if, in addition to a periodic potential, a constant tilting force is applied. Refs. [8, 9] predict that the enhancement of diffusion can be as large as even up to fourteen orders of magnitude under the realistic room temperature conditions. The rapid increase in the effective diffusion coefficient appears due to the interaction between the potential and particles moving at a nonzero mean velocity. This discovery has generated additional interest in the investigation of the Browninan transport pointing to new possibilities in modelling experimental data of stochastic processes in periodic structures. Paper [9] refers to a number of various applications in this context .

A further development in this direction was to apply to the system a sinusoidal time-periodic force in addition to the tilted sinusoidal space-periodic force. It was demonstrated numerically "that the interplay between the frequency-locking and the noise gives rise to a multi-enhancement of the effective diffusion, and a rich behaviour of the current as well, including partial suppression and characteristic resonances" [10], see also Ref. [11] for the effect of the acceleration of diffusion by time-periodic driving. Mathematically exactly the same overdamped Langevin equation with the white thermal noise, sinusoidal spatial potential, sinusoidal time-periodic ac force, and a tilting dc force as in Ref. [10] was considered in paper [12], in which also an analytic solution of the Langevin equation was found, but the authors focused their attention to the influence of a strong ac force rather than of the tilting dc force on the motion of Brownian particles. Another recent development has been to study the influence of the frictional inhomogeneity of the medium to the giant diffusion. Paper [13] considered "a simple minimal model when the potential is sinusoidal and the friction coefficient is also periodic (sinusoidal) with the same period, but shifted in phase". It was demonstrated that both the giant diffusion and the coherence of transport in a tilted periodic potential are sensitive to the frictional properties of the medium. The influence of the space dependent friction on the mobility of an overdamped particle moving in a washboard potential with bias has been studied in [14].

Thus, on the basis of the above-mentioned pioneer investigations one can conclude that in order to obtain a significant amplification of the diffusion of Brownian particles, the following minimal ingredients must be present (i) the thermal noise, (ii) a periodic structure, and (iii) a constant tilting force. Every additional physical agent may bring forward new essential physical features, as was already demonstrated by the inclusion of a force periodic in time

[10, 11] and a periodic friction coefficient [13]. In the present communication we will numerically study the diffusion in a tilted saw-tooth potential on the basis of the general theoretical scheme developed in [8, 9]. The influence of the asymmetry of the potential on the behaviour of the diffusion coefficient will be investigated and the dependencies of the factor of randomness on the tilting force and temperature will be calculated.

The motion of an overdamped Brownian particle under the influence of the periodic potential $V_0(x)$, static external force F and thermal noise is described by the following Langevin equation

$$\eta\dot{x}(t) = -\frac{dV(x)}{dx} + \xi(t), \tag{1}$$

$$V(x) = V_0(x) - Fx, \tag{2}$$

where $V(x)$ is the total deterministic potential, η is the friction coefficient and $\xi(t)$ is the Gaussian white noise with the mean value $\langle\xi(t)\rangle = 0$ and the correlation function $\langle\xi(t)\xi(t')\rangle = 2\eta k_B T\delta(t-t')$.

The general definition of the effective diffusion coefficient can be written as

$$D := \lim_{t\to\infty}\frac{\sigma_x^2(t)}{2t}, \tag{3}$$

where the dispersion of co-ordinate reads

$$\sigma_x^2(t) := \langle x^2(t)\rangle - \langle x(t)\rangle^2. \tag{4}$$

According to Refs. [8, 9] the diffusion coefficient for the model (1), with the periodic boundary conditions imposed, equals ($F \geq 0$)

$$D = D_0\frac{\int\limits_{x_0}^{x_0+L} I_\pm(x)I_+(x)I_-(x)\frac{dx}{L}}{\left[\int\limits_{x_0}^{x_0+L} I_\pm(x)\frac{dx}{L}\right]^3}, \tag{5}$$

where

$$I_+(x) = \frac{1}{D_0}e^{V(x)/k_BT}\int\limits_{x-L}^{x} e^{-V(y)/k_BT}dy, \tag{6}$$

$$I_-(x) = \frac{1}{D_0}e^{-V(x)/k_BT}\int\limits_{x}^{x+L} e^{V(y)/k_BT}dy. \tag{7}$$

Here L is the spatial period of the potential $V_0(x)$, x_0 is an arbitrary point and $D_0 = k_B T/\eta$ is the diffusion coefficient if $V_0(x) = 0$.

The relation between the diffusive and directed components in the Brownian motion can be characterized by the factor of randomness

$$Q := \frac{2D}{L\langle \dot{x} \rangle}. \tag{8}$$

For the model (1) the particle current has the form [8, 9, 15]

$$\langle \dot{x} \rangle = \frac{1 - e^{-LF/k_B T}}{\int\limits_{x_0}^{x_0+L} I_{\pm}(x)\frac{dx}{L}}. \tag{9}$$

From now on we will consider a simple saw-tooth potential V_0 with the amplitude A and the asymmetry parameter k ($0 < k < L$; the value $k = L/2$ corresponding to the symmetric potential).

In order to evaluate the integrals in equations (5)-(7) and (9) at $x_0 = 0$, one needs the expressions of the potential $V_0(x)$ in the following regions

$$V_0(x) = \begin{cases} A\,\frac{k-x}{k}\,, & 0 \le x \le k\,, \\ A\,\frac{k-x}{k-L}\,, & k \le x \le L\,, \\ A\,\frac{L+k-x}{k}\,, & L \le x \le L+k\,, \\ A\,\frac{L+k-x}{k-L}\,, & L+k \le x \le 2L\,. \end{cases} \tag{10}$$

The large enhancement of the diffusion coefficient as a function of the external static force F appears below the critical tilt value F_c which is determined as the threshold for F above which the local minima of $V(x)$ disappear. In the case under consideration

$$F_c = \frac{A}{L-k}. \tag{11}$$

In calculations we choose $L = 1$ and use the dimensionless quantities $\tilde{T} := k_B T/A$, $\tilde{F} := F/F_c$, $\tilde{D} := \eta D/A$, $\tilde{D}_0 := \eta D_0/A = \tilde{T}$, $\langle \tilde{\dot{x}} \rangle := \eta \langle \dot{x} \rangle/A$, omitting in what follows the sign tilde for the sake of brevity.

The effective diffusion coefficient D as a function of the tilting force F at different values of the parameter of asymmetry $k = 0.2, 0.5, 0.8$ is shown in Figs. 1-3, respectively. The function $D(F)$ reveals qualitatively the analogous behaviour as found in Refs. [8, 9]. At the same time, it is seen from the

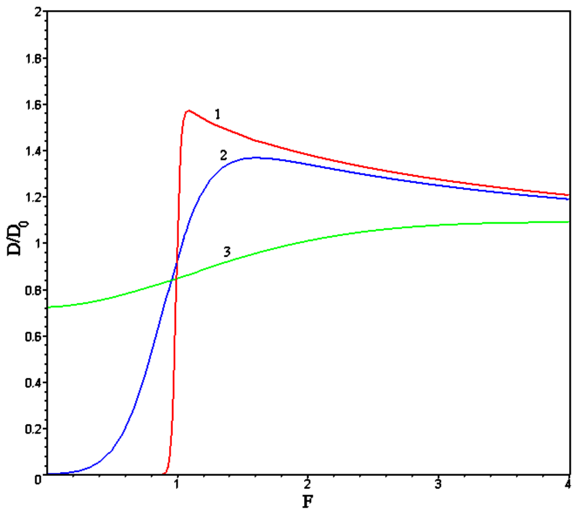

Figure 1: The diffusion coefficient *vs* the tilting force for $k = 0.2$. Curve 1 - $T = 0.01$, curve 2 - $T = 0.1$, curve 3 - $T = 0.5$

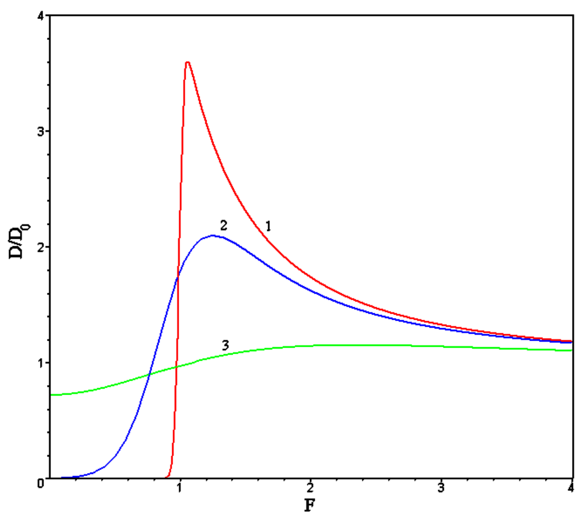

Figure 2: The diffusion coefficient *vs* the tilting force for $k = 0.5$. Curve 1 - $T = 0.01$, curve 2 - $T = 0.1$, curve 3 - $T = 0.5$

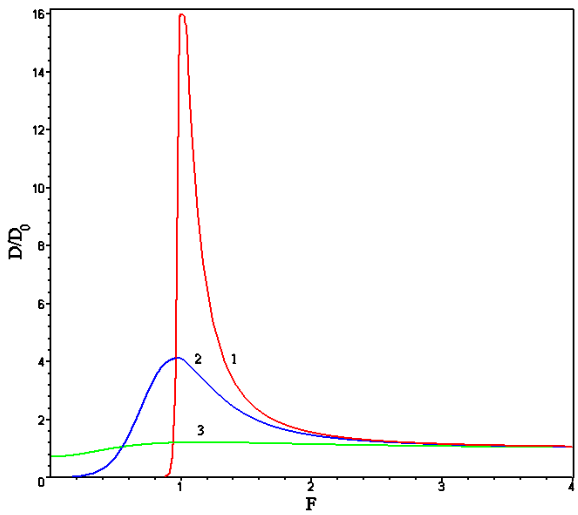

Figure 3: The diffusion coefficient *vs* the tilting force for $k = 0.8$. Curve 1 - $T = 0.01$, curve 2 - $T = 0.1$, curve 3 - $T = 0.5$

comparison of Figs. 1-3 that the large values of k favour the effect of amplification of diffusion with respect to the free diffusion. In the limit $k \to 1$, arbitrary large maximal values of D/D_0 can be obtained, e.g., the diffusion coefficient reaches the value $D_0 \times 10^6$ at $T = 10^{-5}$ and $k = 0.999$. However, if the asymmetry parameter $k \to L$, the critical tilting force $F_c \to \infty$, see equation (11).

On the grounds of the above-mentioned properties of the diffusion and bearing in mind that the tilting force generates the particle current, one can speculate that there exist two interrelated channels for increasing D. The first one is due to the trivial delocalization of Brownian particles as a result of tilting: it is essentially a passive channel. The second channel plays an active role, leading to a really amplified diffusion, which may exceed the free diffusion by many orders of magnitude. The crucial factor here is that there occurs an additional influence of the periodic potential to the Brownian particles which becomes significant if the mean velocity of the particles is sufficiently large. In other words, the character of the interaction between the potential and current obtains qualitatively new features when the current grows strong enough. On the other hand, if the current exceeds a certain critical value determined by the critical tilt, the effect will disappear.

In Fig. 4 we plot the dependencies of the factor of randomness on the tilting. The most interesting feature of the curves in Fig. 4 is the presence of a plateau in the wide region of F up to the critical tilt F_c at a sufficiently low temperature. This means that an extremely neat synchronization of the

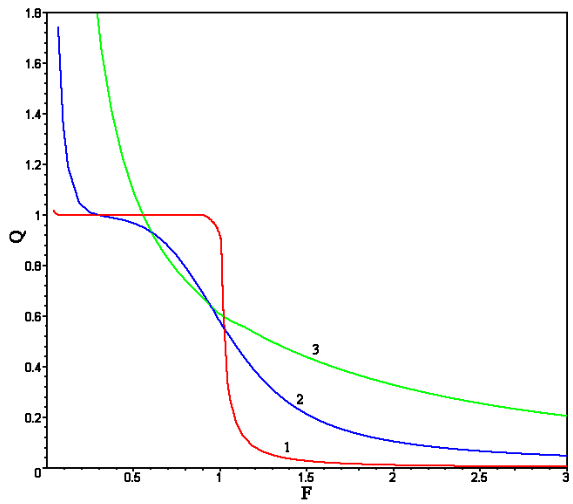

Figure 4: The factor of randomness *vs* the tilting force for $k = 0.5$. Curve 1 - $T = 0.01$, curve 2 - $T = 0.1$, curve 3 - $T = 0.5$

giant enhancements of D and $\langle \dot{x} \rangle$ takes place. Let us emphasize that the plateaus, i.e. the fine tuning of the directed and diffusive motions occur at the values of system parameters where the growth rate of the function $D(F)$ is the largest. At the end of a plateau, $Q(F)$ falls abruptly as F approaches F_c. If $F \to 0$, the current turns to zero and $Q \to \infty$. With the rise of temperature, the length of the plateau diminishes from both sides and the fall near F_c slows down. Finally the plateau disappears and $Q(F)$ decreases monotonically. The calculations show that in the region of low temperatures the variation of the parameter k does not influence the factor of randomness $Q(F)$. For higher temperatures the decrease of the function $Q(F)$ becomes more pronounced if the asymmetry parameter k increases.

The behaviour of the factor of randomness as a function of the temperature is displayed in Fig. 5. It can be seen that at small (curve 1) or at large (curve 7) values of the tilting force F the function $Q(T)$ decreases or increases monotonically. For the intermediate values of F the function $Q(T)$ passes a

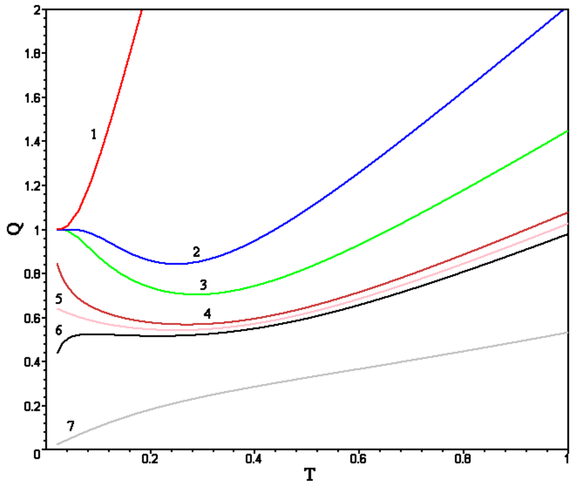

Figure 5: The factor of randomness *vs* temperature for $k = 0.5$. Curve 1 - $F = 0.1$, curve 2 - $F = 0.5$, curve 3 - $F = 0.7$, curve 4 - $F = 0.95$, curve 5 - $F = 1$, curve 6 - $F = 1.05$, curve 7 - $F = 2$

minimum which corresponds to a maximum in the temperature dependence of the Péclet factor Pe$= 2/Q$ (cf. [13]).

To conclude, our results indicate that the shape of a periodic potential (i.e., the asymmetry in our case) essentially influences the amplification of diffusion by a tilting force. We also established that there exists a characteristic synchronization between the diffusion and the current which appears in the region of parameters where the enhancement of $D(F)$ is most rapid. It seems that such a stabilization of the coherence level of the Brownian transport is intrinsically related to the mechanism of the amplification effect of diffusion.

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